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A THEORY FOR THE FORMATION AND PROPAGATION OF LÜDERS BANDS IN A PLATE SUBJECTED TO UNIAXIAL TENSION

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Abstract—It is known from experiments that when a plate, made of a material belonging to a restricted class of elastic-perfectly plastic materials, is subjected to a uniform tensile stress along opposite edges, yielding begins with the formation of Lüders bands. Recent research also shows that if the load is maintained, the bands grow and gradually penetrate the elastic zones.

The study presented here develops a theory for both the formation of the Lüders bands and for the phenomenon of the growth of the plastic zones at the expense of the elastic. No restriction is placed upon Poisson's ratio, though the incompressible material is studied as a special case. We find that for each value of Poisson's ratio there are two possible angles of inclination of the bands. We also establish that when the plastic zone begins to grow for a compressible material the boundary of the plastic zone need not be straight but that the normal vector of this boundary is contained within certain bounds. We also find that the normal velocity at which a point on the boundary propagates depends both on the direction of propagation and on Poisson's ratio.

INTRODUCTION

It is known that for a restricted class of materials, within the general family known as elastic-perfectly plastic materials, yielding in tensile specimens begins when lines or thin bands of slip occur which are inclined to the direction of the applied load. These are the well known Lüders bands. Their formation is attributed to instabilities arising in materials which display an upper yield limit and whose stress level drops somewhat following this initial yield. In the study described here a plate made of a material belonging to this restricted class is subjected to a simple tensile load on opposite edges.

The first part of this study will be directed to establishing the theoretical basis for the formation of Lüders bands, their predicted angle of inclination, and how this angle of inclination changes with changes in Poisson's ratio.

It is also known from experiments that, following the formation of a band, the band will slowly enlarge if the load is maintained. The boundaries of a plastic zone gradually penetrate the elastic domain between bands.

The second part of the paper is a study of the boundary of a plastic zone and the velocity with which it propagates into the elastic domain. It is interesting to find in this study that the velocity with which the boundary of the plastic zone propagates is dependent both on the direction of propagation and on Poisson's ratio.

Neither of the two problems is entirely new; indeed both have been studied by T. Y. Thomas. In his first research into the inclination of Lüders bands [1], Thomas studied a material that is incompressible, that is a material whose Poisson's ratio is one half. In his analysis he used the notion of singular surfaces. Using both the von Mises and Tresca yield criteria, he found the single inclination angle slip to be 35°16′ measured from the line which is perpendicular to the direction of the applied tension. In a second study [2] Thomas relaxed the restriction of incompressibility and investigated the influence of Poisson's ratio on the angle of inclination. He used the constitutive equations due to Henky which, due to their structure, admit a displacement formulation of the problem. He again used both the von Mises and Tresca yield conditions and for each criterion found two slip angles corresponding to each Poisson's ratio.

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In our study of the inclination of Lüders lines we use, as did Thomas, the notion of singular surfaces, but in other respects our development differs somewhat from that of Thomas. We choose to model the material using the Prandtl–Reuss equations rather than those of Henky and use only the von Mises yield criterion. The material we study can have any Poisson's ratio which includes the incompressible as a limiting case. We also find two inclination angles for each value of Poisson's ratio. The first of the two angles for each material is the same as the first of Thomas, whereas the second angle found in our work is different from the second of Thomas. It is not clear why Lüders lines with the second inclination angle have not been observed experimentally.

In his first study[1] Thomas began an investigation of the growth of the plastic zone. However, he restricted his study to a material that is incompressible. Characterizing the plate as an elastic-perfectly plastic Prandtl–Reuss material, he found that the boundaries of the plastic zone can propagate in only two directions, namely in the direction of the applied tension and in the direction perpendicular to it.

In the second part of this paper we study the same problem except that we allow the material to be compressible. We also use the elastic-perfectly plastic Prandtl–Reuss constitutive equations. We find that an incompressible material forms a special case and also, as Thomas did, that the boundaries of the plastic zone propagate either in the direction of the applied tension, or perpendicular to it.

In our study we find that when the material is compressible the boundary of the plastic zone need not be straight. The outward normal vector of the boundary, however, is established as being between certain bounds. The direction of propagation of this boundary, is found to be independent of Poisson's ratio and can vary between two lines inclined at angles of 90° and 35°06′ to the direction of the applied tension. A significant finding in our study of the plastic zone concerns the velocity with which the boundary of the zone propagates. We find that this velocity depends both on the direction of propagation and on Poisson's ratio. To gain insight into the influence of Poisson's ratio on the velocity we dictate that propagation will take place in the direction of the Lüders lines, a direction that is admitted by the bounds, and calculate the velocity of propagation for a range of Poisson's ratios between 0·1 and 0·4.

It is encouraging in light of these findings, to report on experiments conducted in the Department of Material Science at the University of California at Berkeley on a thin plate of steel that displays negligible strain hardening following yield. The experiments show, and record on film, the formation of Lüders bands and the growth of the plastic zone following their formation. It is clear from the film that different parts of boundary of the plastic zone propagate at different velocities and that the velocity depends on the direction of propagation.

1. FORMULATION OF THE PROBLEMS

The plate under study has a thickness which is small compared to the other plate dimensions. Boundaries involving the thickness will be called edges, the other boundaries the faces of the plate. The load is applied as a uniform tensile stress τ along opposite edges of the plate and it is assumed that throughout the plate the stress is uniform through the thickness.

The magnitude of τ is increased until the plate begins to yield. At yield, lines appear on the top and bottom faces of the plate which represent the boundaries of a slip surface which penetrates the thickness of the plate normal to the two faces. A portion of the plate is shown in Fig. 1. The lines AB and CD represent the intersection of slip surfaces with the face of the plate shown.

As a slip surface is treated in each of the two problems as a singular surface, we present here, in the interest of completeness, a brief set of descriptions and definitions to help explain this concept.

When a function is enclosed in square brackets, i.e. $[f]$, the implication is that the function "f" suffers a finite discontinuity across a prescribed surface, which accordingly is called a singular surface. We are concerned in this study with discontinuities of the velocity V_i, and, in particular, with the singular surface across which the normal component of the velocity discontinuity vanishes, i.e. where

$$[V_i]n_i = 0, \tag{1.1}$$

where $\hat{n}$ is the unit normal vector of the singular surface. Such a discontinuity is called a *slip discontinuity*. In equation (1.1) the repeated index, as usual, indicates summation. Physically,

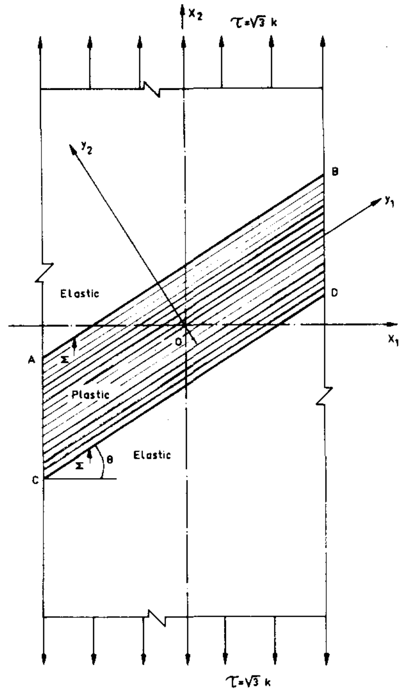

Fig. 1. A portion of plate with plastic slip bands initiated by a uniform tensile load.

equation (1.1) implies that the penetration of the surface by the particles is not allowed so that the discontinuity is due only to the slip of the material particles across the surface.

Such a surface of discontinuity is called a *slip surface* and once slipping begins we are concerned further with the stability of the slip. As our interest here is in slip that is stable we define a singular surface Σ as a surface of stability if every discontinuity $[V_i]$ over Σ is damped out, i.e. if

$$\lim_{t\to\infty}[V_i]=0. \tag{1.2}$$

This behavior can be a consequence of satisfying the equations governing the medium, the boundary conditions and conditions of symmetry.

The material and its behavior, both elastic and plastic, are governed for both problems by the following set of equations.

Equations of motion:

$$\sigma_{ij,j}=\rho\left(\frac{\partial V_i}{\partial t}+V_{i,j}V_j\right), \tag{1.3}$$

conservation of mass:

$$\frac{d\rho}{dt}+\rho V_{i,i}=0, \tag{1.4}$$

the Prandtl–Reuss constitutive equations:

$$\frac{d\sigma_{ij}}{dt}=2\mu\left(\epsilon_{ij}-\frac{3\nu}{E}\frac{dp}{dt}\delta_{ij}-\psi\sigma^*_{ij}\right), \tag{1.5}$$

the von Mises yield condition:

$$\sigma^*_{ij}\sigma^*_{ij}=2k_i^2 \tag{1.6}$$

and the slip condition:

$$[V_i]n_i=0. \tag{1.7}$$

In equations (1.3–1.7), σ_{ij}'s are the components of the stress tensor, ρ is the mass density, σ_{ij}^*'s are the stress deviators defined by

$$\sigma_{ij}^* \equiv \sigma_{ij} + p\delta_{ij}, \tag{1.8}$$

where $p \equiv -\frac{1}{3}\sigma_{kk}$, and ϵ_{ij} are the components of the strain rate tensor

$$\epsilon_{ij} \equiv \frac{de_{ij}}{dt} = \frac{1}{2}(V_{i,j} + V_{j,i}), \tag{1.9}$$

and E, μ and ν are the modulus of elasticity, the shear modulus and Poisson's ratio respectively. The quantity ψ is given by

$$\psi = \frac{\epsilon_{ij}\sigma_{ij}^*}{2k^2}. \tag{1.10}$$

In the above equations subscripts after a comma denote partial differentiation with respect to cartesian coordinates, i.e.

$$\sigma_{ij,j} \equiv \frac{\partial\sigma_{ij}}{\partial x_j}; \quad V_{i,j} \equiv \frac{\partial V_i}{\partial x_j}; \text{ etc.}$$

Use of the ordinary time derivative in place of the covariant time derivative[1, 3] appears to be justified since rotational effects are observed to be of minor significance in our problem.

Because in the development of this study we shall make use of dynamic, geometric, and kinematic conditions of compatibility of first order[1, 4, 5], which are valid over a singular surface, they are as follows:

Dynamic conditions of compatibility:

$$[\rho(V_n - G)] = 0; [\sigma_{ij}]n_j = \rho(V_n - G)[V_i]. \tag{1.11}$$

Geometric conditions of compatibility:

$$\begin{aligned} [V_{i,j}] &= \lambda_i n_j + g^{\alpha\beta}[V_i]_{,\alpha} x_{j\beta}, \\ [\sigma_{ij,k}] &= \xi_{ij} n_k + g^{\alpha\beta}[\sigma_{ij}]_{,\alpha} x_{j\beta}. \end{aligned} \tag{1.12}$$

Kinematic conditions of compatibility:

$$\begin{aligned} \left[\frac{\partial V_i}{\partial t}\right] &= -G\lambda_i + \frac{\delta}{\delta t}[V_i], \\ \left[\frac{\partial \sigma_{ij}}{\partial t}\right] &= -G\xi_{ij} + \frac{\delta}{\delta t}[\sigma_{ij}]. \end{aligned} \tag{1.13}$$

In all the conditions of compatibility, G is the normal component of the propagation velocity of the singular surface Σ, V_n is the normal component of the particle veocity, $g^{\alpha\beta}$ are the contravariant components of the metric tensor of the surface, and the quantities λ_i and ξ_{ij} are suitable functions defined over Σ by

$$\lambda_i \equiv \left[\frac{\partial V_i}{\partial n}\right]; \xi_{ij} \equiv \left[\frac{\partial \sigma_{ij}}{\partial n}\right]. \tag{1.14}$$

Further $x_{i\beta}$'s stand for

$$x_{i\beta} \equiv \frac{\partial x_i}{\partial u^\beta},$$

where u^β are the surface coordinates of Σ whose equation is given by

$$x_i = x_i(u^\beta, t).$$

Throughout the study the range of Greek indices α, β, etc. will be 1 and 2, and that of Latin indices i, j, k, etc. will be 1, 2, and 3. The δ-time derivative which appears in equations (1.13) is defined in [1].

For the two studies that follow, we find it convenient to use two right hand reference frames x_i and y_j (see Fig. 1). The x coordinate system is placed so that the $x_1 - x_2$ plane coincides with the midplane of the plate and the x_2 axis is in the direction of the applied tension. The y coordinate system is obtained by a counterclockwise rotation of the x system about the x_3 axis through an angle θ such that the y_2 axis is perpendicular to the slip plane. Further we let the common origin of the two frames be at the point 0 midway between the slip planes AB and CD.

2. INCLINATION OF SLIP PLANES

In this section all of the quantities will be referred to the y coordinate system. The components of stress and velocity will be designated $\bar{\sigma}_{ij}$ and $\bar{V}_i$ on the plastic side of the slip planes and will be unbarred on the elastic side.

We assume that the external load is applied as a uniformly distributed normal stress on the loading edge and that the plate is large enough compared to the thickness so that we can neglect the effect of the free edges. With these assumptions, the stress field is everywhere uniform through the thickness.

When the stress is initially applied it creates the elastic stress field:

$$\begin{aligned} &\sigma_{11} = \tau \sin^2\theta;\ \sigma_{12} = \tau \sin\theta\cos\theta;\ \sigma_{13} = 0, \\ &\sigma_{22} = \tau\cos^2\theta;\ \sigma_{23} = \sigma_{33} = 0. \end{aligned} \tag{2.1}$$

When the yield limit has been reached by increasing τ, plastic flow begins by the formation of slip planes; the magnitude of τ at which yielding will begin is predicted by a yield condition. In this study we use the von Mises yield condition, equation (1.6), which is appropriate for a tensorial presentation. If the stress field in the elastic region, equation (2.1), is introduced into the von Mises yield condition, the critical value of τ is found to be

$$\tau = \sqrt{3}k. \tag{2.2}$$

Following yielding the stresses in the flow region will, like the elastic stresses, be uniform through the thickness so that the components $\bar{\sigma}_{ij}$ of the stress tensor $\boldsymbol{\sigma}$ will be of the form

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(y_1, y_2, t);\ \bar{\sigma}_{i3} = 0. \tag{2.3}$$

When we recall that we have neglected the influence of the free edges, it is clear that the various quantities which enter into the discussion of the problem, when evaluated on the slip plane, will be independent of the y_1 coordinate. This circumstance leads to what we will call symmetry conditions. For the problem under study they are

$$\bar{\sigma}_{ij,1} = 0;\ \bar{V}_{i,1} = 0. \tag{2.4}$$

On the other hand, for our specific problem the slip condition, equation (1.7), when referred to y system becomes

$$[V_2] = 0,$$

where

$$[V_i] = \bar{V}_i - V_i.$$

Since on the elastic side of the slip plane $V_2 = 0$, we get

$$\bar{V}_2 = 0. \tag{2.5}$$

If we take into account the fact that in our problem the singular surface Σ is stationary, i.e.

$G = 0$, and further that the elastic side of Σ is in equilibrium, i.e. the particle velocities V_i on the elastic side vanish, the dynamic conditions of compatibility reduce to

$$\bar{V}_2 = 0; [\sigma_{i2}] = 0, \quad (2.6)$$

where

$$[\sigma_{ij}] = \bar{\sigma}_{ij} - \sigma_{ij}.$$

We note that the first of equations (2.6) is identical with the slip condition. The second of equations (2.6) can be written as

$$\bar{\sigma}_{i2} - \sigma_{i2} = 0. \quad (2.7)$$

Using plane stress conditions, equations (2.3), if we write the yield condition on the plastic side of Σ we obtain

$$\bar{\sigma}_{11}^2 + 3\bar{\sigma}_{12}^2 + \bar{\sigma}_{22}^2 - \bar{\sigma}_{11}\bar{\sigma}_{22} = 3k^2. \quad (2.8)$$

It is seen that, due to equation (2.7), this can also be written as

$$\bar{\sigma}_{11}^2 + 3\sigma_{12}^2 + \sigma_{22}^2 - \bar{\sigma}_{11}\sigma_{22} = 3k^2, \quad (2.9)$$

or

$$\bar{\sigma}_{11}^2 - (\tau \cos^2\theta)\bar{\sigma}_{11} + \tau^2\cos^4\theta + 3\tau^2\sin^2\theta\cos^2\theta = \tau^2, \quad (2.10)$$

in view of equations (2.1) and equations (2.2). From equation (2.10) it is also seen that $\bar{\sigma}_{11}$ is constant, i.e.

$$\frac{d\bar{\sigma}_{11}}{dt} = 0. \quad (2.11)$$

On the other hand since $\bar{\sigma}_{i2} = \sigma_{i2} = \text{const.}$, from equation (2.7), we should have

$$\frac{d\bar{\sigma}_{22}}{dt} = 0. \quad (2.12)$$

When we write the constitutive equation, equation (1.5), for the plastic side of Σ; for $i = 1, j = 1$ we find

$$\frac{d\bar{\sigma}_{11}}{dt} = 2\mu\left(\bar{\epsilon}_{11} - \frac{3\nu}{E}\frac{d\bar{p}}{dt} - \bar{\psi}\bar{\sigma}_{11}^*\right). \quad (2.13)$$

Further, on account of the symmetry conditions, equation (2.4), we also have

$$\bar{\epsilon}_{11} = 0. \quad (2.14)$$

In view of equations (2.11), (2.12), (2.14), and the definition of p, equation (2.13) reduces to

$$\bar{\psi}\bar{\sigma}_{11}^* = 0. \quad (2.15)$$

There are two ways in which equation (2.15) can be satisfied:

(i) The first way is to have

$$\bar{\sigma}_{11}^* = 0; \bar{\psi} \neq 0, \quad (2.16)$$

which implies

$$\bar{\sigma}_{11} = \tfrac{1}{2}\bar{\sigma}_{22} = \tfrac{1}{2}\sigma_{22} = \tfrac{1}{2}\tau\cos^2\theta. \quad (2.17)$$

Substitution of this expression for $\bar{\sigma}_{11}$ into the yield condition, equation (2.10), gives

$$\tfrac{9}{4}\cos^4\theta - 3\cos^2\theta + 1 = 0, \quad (2.18)$$

from which we have

$$\cos^2\theta = \tfrac{2}{3}, \tag{2.19}$$

from which

$$\theta = \pm 35°16'.$$

We now show that this angle of inclination of slip lines would occur if the material were elastically incompressible. From the constitutive equation, equation (1.5), with the vanishing $\bar{\sigma}^*_{11}$, we have

$$-\frac{d\bar{p}}{dt} = 2\mu\left(\bar{\epsilon}_{11} - \frac{3\nu}{E}\frac{d\bar{p}}{dt}\right), \tag{2.20}$$

or

$$\left(\frac{6\mu\nu}{E} - 1\right)\frac{d\bar{p}}{dt} = 2\mu\bar{\epsilon}_{11}. \tag{2.21}$$

By integrating this equation we get

$$\left(\frac{6\mu\nu}{E} - 1\right)\bar{p} = 2\mu\bar{e}_{11}, \tag{2.22}$$

for the initial stress-free material. But $\bar{e}_{11} = \bar{u}_{1,1} = 0$ in view of the symmetry condition. Therefore

$$\frac{6\mu\nu}{E} - 1 = 0, \quad \text{or} \quad \nu = \frac{1}{2}, \tag{2.23}$$

which characterizes an incompressible elastic material.

(ii) The second way of satisfying equation (2.15) is to have

$$\bar{\sigma}^*_{11} \neq 0; \; \bar{\psi} = 0, \tag{2.24}$$

substitution of which into equation (1.5) yields

$$\frac{d\bar{\sigma}_{ij}}{dt} = 2\mu\left(\bar{\epsilon}_{ij} - \frac{3\nu}{E}\frac{d\bar{p}}{dt}\delta_{ij}\right). \tag{2.25}$$

This is a purely elastic constitutive equation and it implies that all of the strain rate components vanish on the plastic side of the slip surface: i.e. $\bar{\epsilon}_{ij} = 0$. However, it does not follow from this equation that the components vanish elsewhere in the plastic zone. Indeed study of the field equations reveals that $(\partial\bar{\epsilon}_{ij}/\partial y_2) \neq 0$. This shows that not all of the components of the strain rate tensor ϵ_{ij} vanish in the interior of the plastic band even though they are zero just inside the slip surfaces.

If we integrate equation (2.25) we get

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij} - \frac{6\mu\nu}{E}\bar{p}\delta_{ij}, \tag{2.26}$$

for the initial stress-free material. In view of the definition of "p", equation (2.26) can be written as

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij} + \frac{2\mu\nu}{E}\bar{\sigma}_{kk}\delta_{ij}. \tag{2.27}$$

For $i = 1$, $j = 1$, equation (2.27) becomes

$$\bar{\sigma}_{11} = 2\mu\bar{e}_{11} + \frac{2\mu\nu}{E}(\bar{\sigma}_{11} + \bar{\sigma}_{22}), \tag{2.28}$$

or

$$\bar{\sigma}_{11} = \nu\bar{\sigma}_{22} = \nu\sigma_{22} = \nu\tau\cos^2\theta. \tag{2.29}$$

If we substitute the value of $\bar{\sigma}_{11}$ from equation (2.29) into equation (2.10) we get

$$(\nu^2 - \nu - 2)\cos^4\theta + 3\cos^2\theta - 1 = 0, \tag{2.30}$$

which is quadratic in $\cos^2\theta$. Solutions of equation (2.30) are

$$\cos^2\theta = \frac{1}{1+\nu}, \frac{1}{2-\nu}. \tag{2.31}$$

Equation (2.31) suggests that corresponding to any value of Poisson's ratio there exist two inclination angles, say θ_1 and θ_2, for slip bands. Here we note that equation (2.31) holds for the entire range of ν. Hence we write

$$\cos^2\theta_1 = \frac{1}{1+\nu}; \cos^2\theta_2 = \frac{1}{2-\nu}. \tag{2.32}$$

The values of the angles θ_1 and θ_2 are listed in Table 1.

Table 1.

ν	θ_1	θ_2
0.000	± 0°	± 45°
0.100	± 17° 33'	± 43° 32'
0.200	± 24° 6'	± 41° 50'
0.300	± 28° 43'	± 39° 55'
0.400	± 32° 19'	± 37° 48'
0.500	± 35° 16'	± 35° 16'

The effect of compressibility on the inclination of plastic slip bands has been investigated by Thomas[2] using Hencky constitutive equations, which, due to their structure admit a displacement formulation for the problem. In his formulation he used both von Mises and Tresca yield conditions and found also two slip angles corresponding to each Poisson's ratio. Our study differs from that of Thomas in that we use the Prandtl–Reuss equations together with von Mises yield criterion rather than that of Hencky. The inclination angle θ_1 found in this study, equation (2.32) and Table 1, is the same as the first inclination angle obtained by Thomas, based on von Mises yield criterion whereas the angle θ_2 found in our work is somewhat different from the second inclination angle found by Thomas.

Slip along lines inclined at the angle θ_1, has been observed experimentally, but it is not clear why slip along θ_2 lines has not. One of the differences in the two slip planes is established as follows:

Using equations (2.1) and equation (2.29) for $\theta = \theta_1$ we see that

$$[\sigma_{11}] = \bar{\sigma}_{11} - \sigma_{11} = \nu\tau\cos^2\theta_1 - \tau\sin^2\theta_1 = \tau\left(\frac{\nu}{\nu+1} - \frac{\nu}{\nu+1}\right) = 0. \tag{2.33}$$

On the other hand in view of equation (2.7) we also have $\bar{\sigma}_{i2} = \sigma_{i2}$. Therefore we conclude that

$$[\sigma_{ij}] = 0, \tag{2.34}$$

which implies that the stress tensor is continuous over Σ when $\theta = \theta_1$. However, in a similar way it can be shown that $[\sigma_{ij}] \neq 0$ on Σ for $\theta = \theta_2$.

3. PROPAGATION OF THE PLASTIC ZONE

We have found in the previous section that when our plate is subjected to uniform tension, yielding is first manifest as lines of slip called Lüders bands, within which the material is plastically deformed. We are now concerned with changes that take place in the plate, following this initial yielding, when the edge forces are maintained.

The development which follows is based on two postulates. The first is that we can treat the plate as if it were homogeneous even though the bands form domains of inhomogeniety. The second is that, when the tensile stress is maintained following the beginning of yield, the plastic zone will gradually extend from the sides of the band and penetrate into the elastic domain between bands. We learn the nature of the boundary between the two zones, the velocity with which this boundary propagates and the parameters that influence both.

In this section we refer all of the quantities to the x coordinate system. Figure 2 shows Σ^* as the boundary between the plastic zone ABCD and the two elastic zones CDHE and ABFK. At a point on the boundary the unit normal vector is $\hat{n}$ extending from the plastic to the elastic side. The normal component of the velocity with which the boundary is moving at this point has the magnitude G.

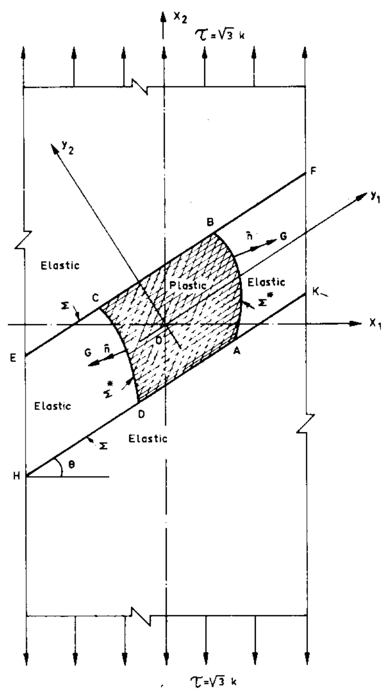

Fig. 2. Propagation of plastic zone into elastic region in a plate subjected to a uniform tensile load.

We begin with a study of the dynamic conditions of compatability, equations (1.11). When applied to the surface Σ^* they can be written

$$\rho G = \bar{\rho}(G - \bar{V}_n),$$
$$[\sigma_{ij}]n_j = -\rho G[V_i]. \tag{3.1}$$

We assume that the boundary Σ^* emerges from the Lüders band which is inclined at the angle θ_1. We make the further assumption, based on the conclusion derived at the end of the last section, that the stress tensor is continuous across Σ^*. Using this assumption of continuity and the fact that $G \neq 0$, we note from the second of equations (3.1) that,

$$[V_i] = 0. \tag{3.2}$$

Hence the velocity is also continuous over the surface Σ^*. As $V_n = 0$, it follows from equation (3.2) that $\bar{V}_n = 0$. Therefore from the first of equations (3.1) we have $\rho = \bar{\rho}$, i.e.

$$[\rho] = 0. \tag{3.3}$$

Because of the continuity of σ_{ij}, V_i, and ρ over Σ^* the geometric and kinematic conditions of compatibility reduce to

$$\begin{aligned} [V_{i,j}] &= \lambda_i n_j, \\ \left[\frac{\partial V_i}{\partial t}\right] &= -G\lambda_i, \\ [\sigma_{ij,k}] &= \xi_{ij} n_k, \\ \left[\frac{\partial \sigma_{ij}}{\partial t}\right] &= -G\xi_{ij}, \end{aligned} \tag{3.4}$$

where λ_i and ξ_{ij} are defined in equations (1.14). In writing equations (3.4) we assumed that Σ^* is a singular surface of order one, which implies that not all of the quantities λ_i and ξ_{ij} vanish over Σ^*[1].

We write the equations of motion on both sides of Σ^* and take the difference, and in view of equation (3.2), we find

$$[\sigma_{ij,j}] = \rho\left[\frac{\partial V_i}{\partial t}\right] + \rho[V_{i,j}]V_j. \tag{3.5}$$

If we take into account the fact that the elastic side of Σ^* is in equilibrium, i.e. $V_j = 0$, and if we use equations (3.4), this equation reduces to

$$\xi_{ij} n_j = -\rho G \lambda_i. \tag{3.6}$$

In the same way we write the Prandtl–Reuss equation, equation (1.5), on both sides of Σ^* and take the difference, which gives

$$\left[\frac{\partial \sigma_{ij}}{\partial t}\right] + [\sigma_{ij,k}]V_k = 2\mu\left([\epsilon_{ij}] - \frac{3\nu}{E}\left[\frac{\mathrm{d}p}{\mathrm{d}t}\right]\delta_{ij} - [\psi\sigma^*_{ij}]\right). \tag{3.7}$$

Using equations (3.4), (1.9) and (3.2), equations (3.7) become

$$-G\xi_{ij} = \mu(\lambda_i n_j + \lambda_j n_i) - \frac{6\mu\nu}{E}\left[\frac{\mathrm{d}p}{\mathrm{d}t}\right]\delta_{ij} - 2\mu[\psi\sigma^*_{ij}]. \tag{3.8}$$

From equation (1.10) and from the continuity of σ_{ij} we can get

$$[\psi\sigma^*_{ij}] = \frac{1}{2k^2}\sigma^*_{ab}\lambda_a n_b \sigma^*_{ij}. \tag{3.9}$$

On the other hand, if we use equations (3.4) and the definition of "p", to compute the jump of

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial p}{\partial t} + p_{,i}V_i \tag{3.10}$$

across Σ^*, we find

$$\left[\frac{\mathrm{d}p}{\mathrm{d}t}\right] = \frac{1}{3}G\xi_{kk}. \tag{3.11}$$

Substitution of equations (3.9) and equation (3.11) into equation (3.8) yields

$$G\xi_{ij} + \mu(\lambda_i n_j + \lambda_j n_i) - \frac{\nu}{\nu+1}G\xi_{kk}\delta_{ij} - \frac{3\mu}{\tau^2}\sigma^*_{ab}\lambda_a n_b \sigma^*_{ij} = 0 \tag{3.12}$$

when use is made of equation (2.2).

We now make use of the fact that our problem is one of plane stress. It follows from equation (1.14) that $\xi_{i3}=0$ and from equation (3.6) that $\lambda_3=0$. Substitution in equation (3.12) and expanding the last term will give

$$\sigma^*_{ab}\lambda_a n_b \sigma^*_{ij} = \tau(\tfrac{2}{3}\lambda_2 n_2 - \tfrac{1}{3}\lambda_1 n_1)(\sigma_{ij} - \tfrac{1}{3}\tau\delta_{ij}). \tag{3.13}$$

Substitution of equation (3.13) into equation (3.12) gives

$$G\xi_{ij} + \mu(\lambda_i n_j + \lambda_j n_i) - \frac{\nu}{\nu+1}G\xi_{kk}\delta_{ij} - \frac{3\mu}{\tau}\left(\frac{2}{3}\lambda_2 n_2 - \frac{1}{3}\lambda_1 n_1\right)\left(\sigma_{ij} - \frac{\tau}{3}\delta_{ij}\right) = 0. \tag{3.14}$$

Putting $i = \alpha(\alpha = 1, 2)$ and $j = 3$ in the equation (3.14) and using plane stress conditions we get

$$\mu\lambda_\alpha n_3 = 0, \tag{3.15}$$

which implies

$$n_3 = 0. \tag{3.16}$$

By multiplying equation (3.14) by n_j and sum over j and using equation (3.6) we obtain

$$(\mu - \rho G^2)\lambda_i - \frac{\nu}{\nu+1}G\xi_{kk}n_i + \mu\left\{\lambda_j n_j n_i - \frac{1}{\tau}(2\lambda_2 n_2 - \lambda_1 n_1)\left(\sigma_{ij}n_j - \frac{1}{3}\tau n_i\right)\right\} = 0. \tag{3.17}$$

When we let $i = j = 3$ in equation (3.14) and solve for ξ_{kk} we get

$$\xi_{kk} = \frac{\mu(\nu+1)}{3\nu G}(2\lambda_2 n_2 - \lambda_1 n_1), \tag{3.18}$$

again exploiting plane stress. We now substitute equation (3.18) into (3.17) to obtain

$$(\mu - \rho G^2)\lambda_i + \mu\left(\frac{4}{3}\lambda_1 n_1 + \frac{1}{3}\lambda_2 n_2\right)n_i - \frac{\mu}{\tau}(2\lambda_2 n_2 - \lambda_1 n_1)\left(\sigma_{ij}n_j - \frac{1}{3}\tau n_i\right) = 0. \tag{3.19}$$

For $i = 3$, equation (3.19) is satisfied identically, substitution of $i = 1$ and $i = 2$ leads to

$$\begin{aligned}(\mu - \rho G^2)\lambda_1 + \mu(\lambda_1 n_1^2 + \lambda_2 n_2 n_1) = 0,\\ (\mu - \rho G^2)\lambda_2 - \mu(\lambda_2 n_2^2 - 2\lambda_1 n_1 n_2) = 0\end{aligned} \tag{3.20}$$

respectively. Equations (3.20) can be written in the matrix form

$$\begin{pmatrix}\rho G^2 - \mu(1+n_1^2) & -\mu n_1 n_2\\ -2\mu n_1 n_2 & \rho G^2 - \mu(1-n_2^2)\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{3.21}$$

Equation (3.21) represents an eigenvalue problem in which the velocity G is an eigenvalue, and λ_1 and λ_2 are the components of the eigenvector. For a nontrivial solution, the determinant of the coefficient matrix must vanish. This circumstance leads to the equation

$$(\rho G^2 - \mu)[\rho G^2 - \mu + \mu(n_2^2 - n_1^2)] - 3\mu^2 n_1^2 n_2^2 = 0. \tag{3.22}$$

When the direction of propagation is specified, equation (3.22) determines the velocities with which the plastic region propagates into the elastic zones. When we make use of the fact that $n_1^2 + n_2^2 = 1$, equation (3.22) can be written in terms of the single component n_1 as

$$\rho^2 G^4 - \rho\mu(1 + 2n_1^2)G^2 + 3\mu^2 n_1^4 - \mu^2 n_1^2 = 0. \tag{3.23}$$

Noting that equation (3.23) is a quadratic equation in G^2, the roots can be written

$$G_{1,2}^2 = \frac{\mu}{2\rho}[(1+2n_1^2) \pm \sqrt{(1+4n_1^2+4n_1^4) - 4n_1^2(3n_1^2-1)}]. \tag{3.24}$$

As the velocities G must be real, the values of G^2 must be both real and positive. The roots will be real if

$$n_1^4 - n_1^2 - \tfrac{1}{8} \leq 0. \tag{3.25}$$

It can be shown that the inequality, equation (3.25), is satisfied for all values of $0 \leq n_1 \leq 1$. Further the roots will be positive only when

$$n_1^2(3n_1^2 - 1) \geq 0. \tag{3.26}$$

The inequality, equation (3.26), will be satisfied if $n_1 = 0$ or $n_1 \geq 1/\sqrt{3}$. Therefore, the velocity G will be real only when

$$n_1 = 0 \quad \text{or} \quad \sqrt{\tfrac{1}{3}} \leq n_1 \leq 1. \tag{3.27}$$

We now consider both of these cases:

(i) $n_1 = 0$. When $n_1 = 0$, $\hat{n} = \hat{x}_2$ which means that the boundary surface is normal to the direction of the applied stress and that propagation is in the direction of the applied stress. To study the conditions that lead to this case we substitute $n_1 = 0$ in equation (3.24) and find that

$$G = \sqrt{\frac{\mu}{\rho}};\ G = 0. \tag{3.28}$$

The corresponding eigenvectors are

$$\boldsymbol{\lambda} = \begin{pmatrix}\lambda_1\\0\end{pmatrix};\ \boldsymbol{\lambda} = \begin{pmatrix}0\\\lambda_2\end{pmatrix} \tag{3.29}$$

respectively. The first solution, which is the only meaningful one, has already been shown to result when the material is incompressible. This is the result already established by Thomas [1].

(ii) $n_1 = 1$, the upper bound of n_1. For this case $n_2 = 0$, $\hat{n} = \hat{x}_1$, so that the boundary is parallel to the direction of the applied stress and the direction of propagation is perpendicular to it.

When we substitute $n_1 = 1$ in equation (3.24) we find

$$G = \sqrt{\frac{\mu}{\rho}};\ G = \sqrt{\frac{2\mu}{\rho}}. \tag{3.30}$$

The corresponding eigenvectors are

$$\boldsymbol{\lambda} = \begin{pmatrix}0\\\lambda_2\end{pmatrix};\ \boldsymbol{\lambda} = \begin{pmatrix}\lambda_1\\0\end{pmatrix} \tag{3.31}$$

respectively. The first solution

$$G = \sqrt{\frac{\mu}{\rho}},\ \boldsymbol{\lambda} = \begin{pmatrix}0\\\lambda_2\end{pmatrix}$$

corresponds as before to the incompressible case.

(iii) $n_1 = \sqrt{1/3}$, the lower bound of n_1. When $n_1 = 1/\sqrt{3}$, $n_2 = \sqrt{2/3}$ so that $\hat{n} = \sqrt{(1/3)}\hat{x}_1 + \sqrt{(2/3)}\hat{x}_2$. This establishes the direction of propagation for this bound as being along a line inclined at 35°06′ to the direction of the applied tension.

For the second case, the material, except at one bound is compressible. To study the influence on the velocity of propagation of changes in Poisson's ratio we study the particular case where propagation is parallel to the first inclination angle θ_1. The value of n_1, dictated by this choice falls

Table 2.

ν	θ_1	G_1^2	G_2^2
0·1	17°33′	$2{\cdot}056\mu/\rho$	$0{\cdot}765\mu/\rho$
0·2	24°06′	$2{\cdot}055\mu/\rho$	$0{\cdot}605\mu/\rho$
0·3	28°43′	$2{\cdot}050\mu/\rho$	$0{\cdot}490\mu/\rho$
0·4	32°19′	$2{\cdot}025\mu/\rho$	$0{\cdot}405\mu/\rho$

within the bounds of the second case given by equation (3.27). As Poisson's ratio changes, so does θ_1, then n_1, resulting in changes in G_1^2 and G_2^2 from equation (3.24). The influence of Poisson's ratio on the propagation velocities is shown in Table 2.

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